

# Anomalous diffusion in convergence to effective ergodicity

Mehmet Süzen\*

*MInstP*

*Asha (Paşaköy) 5561 Famagusta, Cyprus*

(Dated: June 29, 2016)

Power-law exponents in the convergence to effective ergodicity is quantified for Ising-Lenz model in one dimension. Modified Thirumalai-Mountain (TM) metric for magnetisation is computed for the range of temperature values under strongly correlated dynamics. In producing evolution of TM metric over time, time-averaged dynamics is generated by using Metropolis and Glauber single-spin-flip dynamics, and ensemble-averaged dynamics with an exact solution. Superdiffusive behaviour is numerically identified in the parameter regimes studied, i.e., power-law exponents,  $\alpha > 1.0$ .

## I. INTRODUCTION

Dynamics of cooperation among assembly of independent units has been studied in many different context [1], such as model of magnetic material [2–4] or the state of a neuron [5]. Convergence to ergodicity in this kind of cooperation dynamics is explored recently [6] using Thirumalai-Mountain (TM) fluctuation metric [7, 8] that is adapted for Ising model's total magnetization. In this study, we investigate the power-laws emerged from the time evolution of the rate of effective ergodic convergence under the parameters that give rise to strong correlations and anomalous diffusion in coverage. Understanding ergodicity for this circumstances is not only interesting due to fundamental importance in statistical mechanics [9] and anomalous diffusion in general [10], also for its implications in real world applications, such as understanding disruption in neural networks for dementia [11], realisation of associative memory in a solid-state device [12], nature of economic utility [13] and optical lattice dynamics [14].

The Ising model is introduced in Sec. II. In Sec. III, the concept of effective ergodicity will be shortly explained. Measuring convergence to ergodicity is shown in Sec. IV. Computation of power-law will be shown in Sec. V. Summary is given in Sec. VI.

## II. ISING-LENZ SYSTEM

A lattice with  $N$  sites, labelled as  $\{s_i\}_{i=1}^N$ , which can take up two values, such as  $\{1, -1\}$ . These values implies spin up or down in the Ising Model [2–4] or an activation in neuronal systems [5].

The total energy, Hamiltonian of the system can be written with two interactions, one due to nearest-neighbors (NN) and one due to an external field, with coefficients  $J$  and  $H$  respectively,

$$\mathcal{H}(\{s_i\}_{i=1}^N, J, H) = J \left( \sum_{i=1}^{N-1} s_i s_{i+1} + (s_1 s_N) \right) + H \sum_{i=1}^N s_i. \quad (1)$$

the term  $s_1 s_N$  is imposed by the periodic boundary conditions which provides translational invariance, making the model a closed chain of interacting units. Thermal scale is expressed with  $\beta = \frac{1}{k_B T}$  and the corresponding coefficients for NN and external field is scaled by this,

$$K = \beta J, \quad h = \beta H. \quad (2)$$

Analytic expression for finite size magnetisation  $M_E(N, \beta, H)$  is obtained by using Transfer Matrix method. Whereas, time evolution of  $M_E(N, \beta, H)$  is computed via Monte Carlo procedure with Metropolis and Glauber *single-spin-flip dynamics*. Where, more details can be found in the previous work [6].

## III. ERGODIC DYNAMICS

A form of ergodic dynamics suggested by Boltzmann that trajectories of many-body system will reach to phase-space partitions where its likely to be in a thermodynamic equilibrium [9]. At this point, ensemble averages and time averages of the system produce close to identical measures. This implies, for a given observable  $g$  over a fixed phase-space points, ensemble averaged value can be recovered by time average values,  $g(t)$  from  $t_0$  to  $t_N$ ,

$$\langle g \rangle = \lim_{t_N \rightarrow \infty} \frac{1}{t_N - t_0} \int_{t_0}^{t_N} g(t) dt, \quad (3)$$

where  $\langle \rangle$  indicates ensemble averaged value. This kind of basic definition is not standard in the literature [7, 15, 16]. Other forms of *ergodicity* demands that system should visit all available phase-space partitions, which might not be possible for a finite physical system,

---

\* mehmet.suzen@physics.org

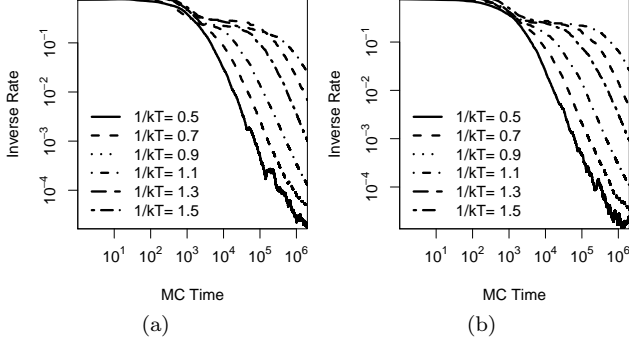


FIG. 1. Effective ergodic convergence in Metropolis and Glauber dynamics for  $N = 512$  with varying temperature and fixed external field  $H = \{1.0\}$  at (a) and (b) respectively.

and moreover feasibility of this type of *ergodicity* is questioned recently [17]. In practical terms, since partitions of the phase-space is clustered around few regions, *effective ergodicity* can be attained quickly [7].

On the other hand, *ergodicity* for the *single-spin-flip dynamics*, essentially a Markov process, defined as accessibility of any given state point to an other state point over time [16, 18]. In this sense, Monte Carlo procedure used in this work is ergodic by construction for long enough times.

#### IV. MEASURING CONVERGENCE

*Effective ergodic convergence*,  $\Omega_G(t)$  for a given observable  $g$ , can be constructed based on the fact that identical parts of cooperating units have identical characteristics at thermodynamic equilibrium [7]. This is realised by Thirumalai-Mountain (TM)  $G$ -fluctuating metric [7, 8], at a given time  $t_k$

$$\Omega_G(t_k) = \frac{1}{N} \sum_{j=1}^N [g_j(t_k) - \langle g(t_k) \rangle]^2, \quad (4)$$

where time average per unit expressed as  $g_j(t_k)$  and  $\langle g(t_k) \rangle$  is the instantaneous ensemble average over all units,

$$g_j(t_k) = \frac{1}{k} \sum_{i=0}^k g_j(t_i), \quad (5)$$

$$\langle g(t_k) \rangle = \frac{1}{N} \sum_{j=1}^N g_j(t_k). \quad (6)$$

Consequently, the rate of ergodic convergence is measured as

$$\Omega'_G = \frac{\Omega_G(t)}{\Omega_G(0)} \rightarrow \frac{1}{tD_G} \quad (7)$$

where diffusion coefficient expressed as  $D_G$ . Similar measure of ergodicity is used in simple liquids [7, 19], and earthquake fault networks [20, 21].

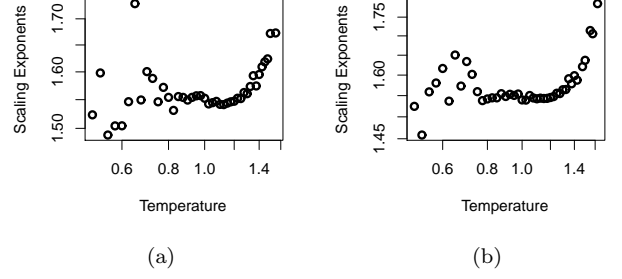


FIG. 2. Power-law exponents in effective ergodic convergence in Metropolis and Glauber dynamics for  $N = 512$  with varying temperature ( $\beta$ ) and fixed external field  $H = \{1.0\}$  at (a) and (b) respectively. Sharp increase at around  $\beta = 1.4$ .

Using  $\Omega_G$  for total magnetization in Ising Model at time  $t_k$  as a function of temperature and external field values reads

$$\Omega_M(t_k, N, \beta, h) = [M_T(t_k) - M_E]^2, \quad (8)$$

$$M_T = \frac{1}{k} \sum_{i=0}^k M(t_i), \quad (9)$$

where  $M_T(N, \beta, h)$  and  $M_E(N, \beta, h)$  correspond to time and ensemble averaged total magnetization. Note that exact expression for  $M_E$  is used [6]. Hence in computing  $\Omega_M(t_k, N, \beta, h)$ , we generate Metropolis and Glauber single-spin-flip dynamics for  $N = 512$  cooperating units, a range of Boltzmann factor values  $\beta = [0.5, 1.5]$  with a spacing of 0.025 and non-zero external field values  $H = 1.0$  with setting short-range interaction strength to  $J = 1.0$  [22]. Time evolution of inverse rate,  $\Omega_G^{-1}$  is given in Figure 1(a) and Figure 1(b). After an initial period, inverse rate value decreases rapidly. At higher temperatures, inverse rate behaves differently, we quantify this using power-laws.

#### V. POWER LAWS

Inverse rate  $\Omega_G'^{-1}$  in the effective ergodic measure scales with Monte Carlo time in a power-law fashion,

$$\Omega_G'^{-1} \sim t^{-\alpha}$$

where  $\alpha$  is the the exponent and  $t$  is the Monte Carlo time. While, power laws in complex systems and computing them on a given empirical data are studied in depth [23, 24]. We compute power-law exponents of evolution of inverse rate over Monte Carlo time for the parameters given at Figure 1(a) and 1(b). Identification where to start to compute  $\alpha$  is achieved by Kolmogorov-Smirnov test [25]. Temperature dependence of  $\alpha$  is reported in Figure 2(a) and Figure 2(b), where  $\alpha > 1.0$  is observed

leading to superdiffusive behaviour. We also observe that there is a sharp increase for  $\alpha$  at around  $\beta = 1.4$ , along with a stable value in the range  $\beta = (0.8, 1.4)$ .

## VI. SUMMARY

Understanding how ergodicity is reached empirically plays a vital role both theoretically and from practical standpoint. We have identified superdiffusive behaviour and power-law exponents for the rate of ergodic convergence in Ising Model under parameters give rise

to strong correlation, using the modified Thirumalai-Mountain (TM) metric for the total magnetization. We conclude that combination of stronger temperature values give rise to lower exponents, attributed to high external field.

## ACKNOWLEDGEMENTS

The author would like to dedicate this work for the memory of Michael Kassialos, a folk artist from Asha, Cyprus.

- 
- [1] G. H. Wannier. The statistical problem in cooperative phenomena. *Rev. Mod. Phys.*, 17:50–60, Jan 1945.
  - [2] Ernst Ising. Beitrag zur theorie des ferromagnetismus. *Zeitschrift für Physik A Hadrons and Nuclei*, 31(1):253–258, 1925.
  - [3] Stephen G Brush. History of the lenz-ising model. *Reviews of Modern Physics*, 39(4):883, 1967.
  - [4] RJ Baxter. *Exactly solvable models in statistical mechanics*. Academic Press London, 1982.
  - [5] John J Hopfield. Neural networks and physical systems with emergent collective computational abilities. *Proceedings of the national academy of sciences*, 79(8):2554–2558, 1982.
  - [6] M. Süzen. Effective ergodicity in single-spin-flip dynamics. *Physical Review E*, 90(3):032141, 2014.
  - [7] Raymond D Mountain and D Thirumalai. Measures of effective ergodic convergence in liquids. *The Journal of Physical Chemistry*, 93(19):6975–6979, 1989.
  - [8] D. Thirumalai, R.D. Mountain, and T.R. Kirkpatrick. Ergodic behavior in supercooled liquids and in glasses. *Physical Review A*, 39(7):3563, 1989.
  - [9] Jay Robert Dorfman. *An introduction to chaos in nonequilibrium statistical mechanics*. Number 14. Cambridge University Press, 1999.
  - [10] Ralf Metzler, Jae-Hyung Jeon, Andrey G Cherstvy, and Eli Barkai. Anomalous diffusion models and their properties: non-stationarity, non-ergodicity, and ageing at the centenary of single particle tracking. *Physical Chemistry Chemical Physics*, 16(44):24128–24164, 2014.
  - [11] RA Thuraishingham. Dementia and hopfield model. *Journal of Neural Transmission*, 122(6):773–777, 2015.
  - [12] SG Hu, Y Liu, Z Liu, TP Chen, JJ Wang, Q Yu, LJ Deng, Y Yin, and Sumio Hosaka. Associative memory realized by a reconfigurable memristive hopfield neural network. *Nature communications*, 6, 2015.
  - [13] Ole Peters and Murray Gell-Mann. Evaluating gambles using dynamics. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 26(2):023103, 2016.
  - [14] Michael Schreiber, Sean S Hodgman, Pranjal Bordia, Henrik P Lüschen, Mark H Fischer, Ronen Vosk, Ehud Altman, Ulrich Schneider, and Immanuel Bloch. Observation of many-body localization of interacting fermions in a quasirandom optical lattice. *Science*, 349(6250):842–845, 2015.
  - [15] SK Ma. *Statistical mechanics*. World Scientific, Singapore, 1985.
  - [16] JFC Kingman. The ergodic behaviour of random walks. *Biometrika*, pages 391–396, 1961.
  - [17] Bernard Gaveau and Lawrence S Schulman. Is ergodicity a reasonable hypothesis for macroscopic systems? *The European Physical Journal Special Topics*, 224(5):891–904, 2015.
  - [18] AG Pakes. Some conditions for ergodicity and recurrence of markov chains. *Operations Research*, 17(6):1058–1061, 1969.
  - [19] Vanessa K de Souza and David J Wales. Diagnosing broken ergodicity using an energy fluctuation metric. *The Journal of Chemical Physics*, 123(13):134504, 2005.
  - [20] KF Tiampo, JB Rundle, W Klein, JS Sá Martins, and CD Ferguson. Ergodic dynamics in a natural threshold system. *Physical Review Letters*, 91(23):238501–238501, 2003.
  - [21] KF Tiampo, JB Rundle, W Klein, J Holliday, JS Sá Martins, and CD Ferguson. Ergodicity in natural earthquake fault networks. *Physical Review E*, 75(6):066107, 2007.
  - [22] M. Süzen. *isingLenzMC: Monte Carlo for classical Ising Model*, 2015. v0.2.3 <http://cran.r-project.org/web/packages/isingLenzMC/index.htm>
  - [23] Mark EJ Newman. Power laws, pareto distributions and zipf’s law. *Contemporary physics*, 46(5):323–351, 2005.
  - [24] Aaron Clauset, Cosma Rohilla Shalizi, and Mark EJ Newman. Power-law distributions in empirical data. *SIAM review*, 51(4):661–703, 2009.
  - [25] Gabor Csardi and Tamas Nepusz. The igraph software package for complex network research. *InterJournal, Complex Systems*, 1695(5):1–9, 2006.